Scaling and infrared divergences in the replica field theory of the Ising spin glass

T. Temesvári[1,a], C. De Dominicis[2], and I. Kondor[3]

[1] Institute for Theoretical Physics, Eötvös University, Pázmány Péter sétány 1, Bld. A, 1117 Budapest, Hungary
[2] Service de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette Cedex, France
[3] Department of Physics of Complex Systems, Eötvös University, Pázmány Péter sétány 1, Bld. A, 1117 Budapest, Hungary

Abstract. Replica field theory for the Ising spin glass in zero magnetic field is studied around the upper critical dimension $d = 6$. A scaling theory of the spin glass phase, based on Parisi's ultrametrically organised order parameter, is proposed. We argue that this infinite step replica symmetry broken (RSB) phase is nonperturbative in the sense that amplitudes of scaling forms cannot be expanded in term of the coupling constant w^2. Infrared divergent integrals inevitably appear when we try to compute amplitudes perturbatively, nevertheless the ϵ-expansion of critical exponents seems to be well-behaved. The origin of these problems can be traced back to the unusual behaviour of the free propagator having two mass scales, the smaller one being proportional to the perturbation parameter w^2 and providing a natural infrared cutoff. Keeping the free propagator unexpanded makes it possible to avoid producing infrared divergent integrals. The role of Ward-identities and the problem of the lower critical dimension are also discussed.

PACS. 75.10.Nr Spin-glass and other random models – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.)

Spin glasses [1] have posed a formidable theoretical challenge ever since Edwards and Anderson [2] proposed their simple looking model containing the two basic features of disordered systems: randomness and frustration. Sherrington and Kirkpatrick (SK) [3] defined the mean field version of this model on a fully-connected lattice where each spin interacts with all the others. The highly non-trivial solution of this mean field model by Parisi [4] is now generally accepted as the correct one.

There is, however, no consensus on whether the picture of ultrametrically organised pure states emerging from Parisi's solution survives in finite dimensional, short range systems. "Droplet theory" [5–7] claims the opposite: in d dimensions the equilibrium spin glass phase is unique, apart from reflections of the Ising spins, and a transition from the paramagnet to the spin glass takes place only in zero magnetic field. A huge amount of numerical work has been devoted to clarifying this point (see [8] and references therein) and, in contrast to droplet theory, a complex phase space structure and ultrametricity seem to emerge in four dimensions. The interpretation of recent rigorous results of mathematical physics [9,10] is ambiguous, and what we can learn from them is that even the definition of some quantities like the probability distribution of overlaps, $P(q)$, is a difficult problem, and that chaotic size dependence can render the thermodynamic

limit meaningless in some cases. We feel that our findings below about the unavoidable infrared problems of replica field theory, arising from a blind application of perturbation expansion, are somehow related to these phenomena.

It is clear that analytical methods are important to settle this decade-long debate. One way to depart from the SK limit is the $1/d$ expansion of Georges *et al.* [11] supporting the survival of the replica symmetry broken (RSB) picture of Parisi in high spatial dimensions. Another important question is whether a spin glass transition exists in a nonzero magnetic field. Droplet theory gives a negative answer, while the onset of RSB at the de Almeida-Thouless (AT) [12] line is an important feature of Parisi's mean field theory. Bray and Roberts [13] have studied the problem using Wilson's renormalization group in the replica symmetric (RS) high-temperature phase, constraining the fluctuating fields into the replicon subspace. They did not find any meaningful fixpoint, thus suggesting that there is no AT-line around 6 dimensions. We feel, however, that their projected field theory, with all the masses other than that of the replicon taken to be infinite, is not sufficient to settle the question since even the crossover region including the known zero-field fixed-point [14] is impossible to reach in this model with only two cubic coupling constants. Taking into account all the eight cubic invariants of the permutation group we would have a large enough parameter space with eight φ^3

[a] e-mail: `temtam@poe.elte.hu`

coupling constants and three masses. A renormalisation study, similar to that of Bray and Roberts, of this model could resolve this problem (work is in progress in this direction).

In this paper we study the replica field theory corresponding to the spin glass phase of the Edwards-Anderson model in zero magnetic field and just below the transition point. We are mostly interested in systems with spatial dimensions around the upper critical dimension $d_{\mathrm{u}} = 6$, the case of higher dimensions $d > 8$ and the problem of the lower critical dimension d_{l} are left to the end. We have made an extensive perturbation analysis of the equation of state and found infrared divergent terms at two-loop order for $d < 6$. We will point out that it is the small mass of the bare propagator, rather than the zero (Goldstone) modes, that is responsible for these singularities: it is proportional to w^2, the perturbation parameter. When physical quantities are expanded in terms of w^2, the natural infrared cutoff of the theory is destroyed. Nevertheless, below six dimensions, the full theory has only one mass scale, and we propose a simple scaling theory where all the infinities coming from the infrared integrals are absorbed into the amplitudes of scaling forms.

Before going into details, we would like to mention that a lot of effort has been devoted to the difficult problem of infrared divergences also by Parisi and his collaborators [15,16]. They have proposed a possible cancellation mechanism decreasing the degree of infrared singularities in the replica field theory of the spin glass phase.

As explained in reference [17], bare coupling constants higher than cubic order must be set to zero when studying the critical region. The Lagrangean can be split into a Gaussian and an interaction part:

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(\mathrm{I})},$$

with

$$\mathcal{L}^{(2)} = \frac{1}{2} \sum_{\alpha<\beta,\gamma<\delta} \sum_{\mathbf{p}} \varphi_{\mathbf{p}}^{\alpha\beta} \left(\tilde{G}^{-1}(p) \right)_{\alpha\beta,\gamma\delta} \varphi_{-\mathbf{p}}^{\gamma\delta},$$

where

$$\left(\tilde{G}^{-1}(p) \right)_{\alpha\beta,\alpha\beta} = p^2 - r_0,$$
$$\left(\tilde{G}^{-1}(p) \right)_{\alpha\gamma,\beta\gamma} = -wQ_{\alpha\beta}, \tag{1}$$

and all the other components of $\tilde{G}^{-1}$ are zero. r_0 and w are the bare mass and cubic coupling constant, respectively, while $Q_{\alpha\beta}$ is the *exact* order parameter matrix. Greek indices above and in the following stand for replica numbers going from 1 to n, and n is set to zero, according to the replica trick, at the very end of the calculations. The above Lagrangean is defined as a functional of the fields $\varphi_{\mathbf{p}}^{\alpha\beta}$ with zero average values:

$$\langle \varphi_{\mathbf{p}}^{\alpha\beta} \rangle = 0. \tag{2}$$

(Statistical averages $\langle \ldots \rangle$ are calculated with the weight $\sim \mathrm{e}^{\mathcal{L}}$.) Equation (2) expresses the shift of $\varphi^{\alpha\beta}$, fluctuating

now around the exact $Q_{\alpha\beta}$, as usual in the description of symmetry broken phases. $Q_{\alpha\beta}$ enters also $\mathcal{L}^{(\mathrm{I})}$:

$$\mathcal{L}^{(\mathrm{I})} = \mathcal{L}^{(1)} + \mathcal{L}^{(3)}$$

with

$$\mathcal{L}^{(1)} = \sqrt{N} \sum_{\alpha<\beta} \varphi_{\mathbf{p}=\mathbf{0}}^{\alpha\beta} \left[r_0 Q_{\alpha\beta} + w(Q^2)_{\alpha\beta} \right],$$

$$\mathcal{L}^{(3)} = \frac{1}{\sqrt{N}} w \sum_{\alpha<\beta<\gamma} \sum_{\mathbf{p_1},\mathbf{p_2}} \varphi_{\mathbf{p_1}}^{\alpha\beta} \varphi_{\mathbf{p_2}}^{\beta\gamma} \varphi_{-(\mathbf{p_1}+\mathbf{p_2})}^{\gamma\alpha}.$$

(N, the number of sites of the d-dimensional hypercubic lattice, is taken to infinity in the thermodynamic limit.)

We concentrate on the determination of $Q_{\alpha\beta}$ near T_{c} in zero magnetic field where the above Lagrangean is taken with $r_0 = r_0^{(\mathrm{c})} + r$, $r \ll 1$. For such a cubic field theory, equation (2) can be written as a closed system of two equations:

$$rQ_{\alpha\beta} + w(Q^2)_{\alpha\beta} = - \left[w \int^{\Lambda} \sum_{\rho \neq \alpha,\beta} \hat{G}_{\alpha\rho,\beta\rho}(p) + r_0^{(\mathrm{c})} Q_{\alpha\beta} \right]$$
$$\equiv -Y_{\alpha\beta}, \tag{3}$$
$$\langle \varphi_{\mathbf{p}}^{\alpha\beta} \varphi_{-\mathbf{p}}^{\gamma\delta} \rangle = \hat{G}_{\alpha\beta,\gamma\delta}(p), \tag{4}$$

where we used the shorthand notation $\int^{\Lambda} \equiv \int^{\Lambda} \frac{\mathrm{d}^d p}{(2\pi)^d}$. $\hat{G}$ is the exact propagator, while the critical value of r_0 is to be determined from the implicit equation:

$$r_0^{(\mathrm{c})} = - \lim_{Q \to 0} \frac{1}{Q_{\alpha\beta}} w \int^{\Lambda} \sum_{\rho \neq \alpha,\beta} \hat{G}_{\alpha\rho,\beta\rho}^{(\mathrm{c})}(p), \tag{5}$$

with $\lim_{Q \to 0}$ meaning that all the elements of Q go to zero. $\hat{G}^{(\mathrm{c})}$ depends on Q and $r_0^{(\mathrm{c})}$ in a complicated manner. By equation (5), $r_0^{(\mathrm{c})}$ is a function of the coupling constant w, and it can be calculated perturbatively.

Computing $Q_{\alpha\beta}$ from equations (3, 4) is a difficult problem, as it is for any nontrivial field theory. Without assuming an ansatz for the structure of $Q_{\alpha\beta}$, it is completely hopeless to find even an approximate solution. We can try, however, to follow the traditional way by starting from a mean field-like equation whose solution is taken as a zeroth order approximation in a systematic perturbative treatment. The most obvious way of defining a mean field equation of state from equation (3) is to drop the loop term $Y_{\alpha\beta}$. Any solution found must then be submitted to the test of stability: the mass operator $\Gamma \equiv \hat{G}^{-1}(\mathbf{p} = 0)$ should not have negative eigenvalues. The illusion created by the marginally stable replica symmetric mean field solution is dispelled by calculating the first loop correction (this was made in Ref. [18]): Bray and Moore found that the hitherto zero, so called replicon, eigenvalue moves to a negative value, a clear indication of an effective quartic coupling (we call it $\tilde{u}$) generated by the cubic coupling at the first loop level [17].

For a meaningful mean field theory, we have to keep the loop term $Y_{\alpha\beta}$ in equation (3) and make a reasonable truncation in equation (4). We can avoid instability by assuming an infinite step RSB structure for $Q_{\alpha\beta}$. It turns out that in this case $Q_{\alpha\beta}$ is highly insensitive to the details of the propagators in the far infrared region, while the near infrared propagators which determine $Q_{\alpha\beta}$ are not influenced by the quartic couplings (for a discussion of the behaviour of the propagator components in the far and near infrared, corresponding to the two mass scales, see Ref. [17]). We have therefore a freedom in truncating equation (4), we can even completely neglect $\mathcal{L}^{(\mathrm{I})}$ when computing the average, thus taking $\hat{G} = \tilde{G}$. Alternatively, we may take into account the quartic couplings, $\tilde{u}$ for instance, being generated from the interaction part. In any case, the propagators are the same in the near infrared, while only the amplitudes are influenced in the far infrared region giving a relatively moderate divergence $\hat{G}_{\alpha\rho,\beta\rho} \sim p^{-3}$. The role of the small mass is to provide an infrared cutoff, and our mean field theory is certainly meaningful for $d > 3$.

Using the tables from reference [17] for the near infrared propagators, we can approximately compute the Parisi order parameter function in the vicinity of T_{c}:

$$Q_{\alpha\beta} = Q(x) = \frac{w}{2\tilde{u}}x, \qquad x = \alpha \cap \beta < x_1, \qquad (6)$$

$$Q(x) = Q_1 = \frac{1}{2w}r, \qquad x > x_1,$$

$$x_1 = \frac{\tilde{u}}{w^2}r, \qquad (7)$$

where Q_1 and x_1 stand for the plateau value and breakpoint of $Q(x)$, respectively. The effective quartic coupling is [17,19]:

$$\tilde{u} = 12w^4 \int^{\Lambda} \frac{1}{p^4(p^2 + r)^2} \cdot \qquad (8)$$

$Q(x)$ in equation (6) is very much like that of the SK model, where the coupling constants w and $\tilde{u}$ are 1. We can learn further details about $Q(x)$ by writing a scaling form for $Y_{\alpha\beta} = Y(x)$. Considering the two mass scales of the mean field propagators

$$m_{\mathrm{large}}^2 \sim r,$$

$$m_{\mathrm{small}}^2 \sim \frac{\tilde{u}}{w^2}r^2 = x_1 r,$$

we have in the critical region:

$$Y(x) = w \int^{\Lambda} \frac{1}{p^2} f_{\mathrm{mf}}\left(\frac{p^2}{r}, \frac{p^2}{x_1 r}, \frac{x}{x_1}\right), \qquad (9)$$

where f_{mf} is a scaling function. The truncation procedure for the propagator has an effect on f_{mf} in the momentum range around the small mass ($p^2 \sim x_1 r$), but the near infrared region ($p^2 \sim r$) and the power of the far infrared limit ($\sim p^{-1}$) are not influenced by it.

The mean field approximation presented above is highly nontrivial, and the order parameter function depends in a very complicated manner on the perturbation parameter w^2. The origin of this complicated dependence can be understood from the fact that the small mass is proportional to w^2:

$$m_{\mathrm{small}}^2 \sim \frac{\tilde{u}}{w^2} \sim w^2 \qquad (10)$$

giving rise to a w-dependence of the propagator $\hat{G}$. At this point we are tempted to compute $Q(x)$ by perturbation expansion, *i.e.* order by order in w^2. This, however, inevitably leads to infrared divergent terms since the infrared cutoff provided by the small mass is then destroyed. For example, in the expansion of Q_1 we have a third order term like

$$wQ_1 \sim w^6 r^{d-4} \int^{\Lambda} \frac{1}{p^8}$$

which is infinite for $d < 8$. Similarly, as we will see later, when using simple-minded perturbation expansion for the full theory we may encounter infrared divergent infinite terms that can, however, build up an infrared cutoff, thus rendering the theory well-behaved above the lower critical dimension.

Following standard arguments of the theory of critical phenomena, we expect a qualitative agreement between full and mean field theory above the upper critical dimension ($d > 6$). Near T_{c}, $Y(x)$ ($x = \alpha \cap \beta$) in equation (3) can be written as:

$$Y(x) = w \int^{\Lambda} \frac{1}{p^{2-\eta}} f\left(\frac{p^2}{r^{2\nu}}, \frac{p^2}{x_1 r^{2\nu}}, \frac{x}{x_1}\right), \quad 6 < d < 8, \quad (11)$$

where the scaling function f depends now on the coupling w and, for later reference, the mean field exponents $\eta = 0$, $\nu = 1/2$ were included. A renormalised $\tilde{u}$ is now understood in equation (7) for the breakpoint of equation (11), preserving its temperature scaling

$$x_1 \sim r^{\frac{d}{2}-3}, \qquad 6 < d < 8. \qquad (12)$$

Going below the upper critical temperature seems to be easy now: the exponents η and ν in equation (11) are then nontrivial and depend on the dimension (they have been computed from the critical theory up to $O(\epsilon^3)$, $\epsilon = 6 - d$, in Ref. [14]). The most important point to clarify is the role of x_1 in the scaling of the $Y(x)$ term. Equation (12) suggests that x_1 becomes temperature independent in $d = 6$. We have made a rather difficult calculation of the $O(\epsilon)$ correction to the temperature exponent of x_1 (see later) and found that it is zero. The hypothesis of a constant, nonuniversal, *i.e.* temperature independent and w, Λ dependent, x_1 is strongly suggested by this finding. The simple scaling picture restored in $d = 6$ would then survive even below the upper critical dimension and $Y(x)$ would take the form

$$Y(x) = w \int^{\Lambda} \frac{1}{p^{2-\eta}} \bar{f}\left(\frac{p^2}{r^{2\nu}}, x\right), \qquad d < 6, \qquad (13)$$

with the nonuniversal $\bar{f}$ replacing f in equation (11).

Provided that our assumption of a temperature independent x_1 is correct, a simple form for the order parameter function is strongly suggested[1]:

$$Q(x) = r^\beta \bar{Q}(x), \qquad (14)$$

where $\bar{Q}(x)$ depends only on w and Λ and for the critical exponent β we have the scaling law

$$\beta = \frac{1}{2}\nu(d - 2 + \eta). \qquad (15)$$

We can find full consistency between equations (13, 14) by assuming the limiting form

$$\bar{f}(v, x) \sim v^{-\frac{1+\beta}{2\nu}}, \qquad v \gg 1 \qquad (16)$$

and using the scaling law equation (15). Changing the integration variable from $\mathbf{p}$ to $\mathbf{p}' = \mathbf{p}/r^\nu$:

$$Y(x) = r^{2\beta} w \int^{\Lambda/r^{2\nu}} \frac{1}{p'^{2-\eta}} \bar{f}(p'^2, x)$$
$$\cong r^{1+\beta} Y_1(x) + r^{2\beta} Y_2(x),$$

where the first term is the leading contribution coming from the upper limit of the integration while the second one is from the bulk. A $Q(x)$ as in equation (14) is then obviously consistent with equation (3).

The above theory for the spin glass phase below $d = 6$, despite its consistency and its clear connection to mean field theory, does require some analytical support. We have made a detailed perturbative computation of the equation of state (3) and our findings can be summarized in the following points:

1. A Parisi RSB scheme is building up order by order with a polynomial $Q(x) = ax + cx^3 + \dots$. Consequently, $\bar{f}(v, x)$ of equation (13) must be also a polynomial in x for any fixed, nonzero v.
2. We computed several logarithms of r to check the above scaling theory and found agreement with calculations of the critical exponents in the critical (massless) phase [14]. The most important result comes from the second order (two-loop) coefficient of $x^3 \log r$ that provides the temperature exponent of the slope a of $Q(x)$ showing that it is equal to β, at least up to $O(\epsilon)$. This is consistent with the scaling form in equation (14) and supports the basic assumption of a temperature-independent x_1.
   The exponent β is originally defined by

$$Q_1 = Cr^\beta \qquad (17)$$

and the one-loop $x \log r$ term gives β correctly up to $O(\epsilon)$ ($\beta = 1 + \frac{\epsilon}{2}$). Furthermore, using the two-loop $x \log^2 r$ results, exponentiation was checked and agreement with the known fixed-point value obtained [14] (the latter result has been published in [17]).

[1] This factorization of r^β from $Q(x)$ was raised by Young some years ago in private communications.

3. We know from exact Goldstone-theorem-like considerations [20] that an infinite-step RSB solution of the equation of state, equations (3, 4), must be marginally stable since the most dangerous replicon eigenvalues are Goldstone-modes. In contrast to the RS case [18], the one-loop calculation gives indeed a marginally stable spectrum of the mass operator.
4. The above three points support a Parisi-type structure of $Q_{\alpha\beta}$, but now we have to deal with the problem of the unremovable infrared divergences appearing when a blind expansion in term of w^2 is used. Up to second order, $Y_{\alpha\beta}$ can be written as

$$Y_{\alpha\beta} = E_{\alpha\beta} + F_{\alpha\beta} + O(w^5), \qquad (18)$$

where

$$E_{\alpha\beta} \equiv w \int^\Lambda \sum_{\rho \neq \alpha,\beta} G_{\alpha\rho,\beta\rho}(p) + r_0^{(c)} Q_{\alpha\beta}, \qquad (19)$$

$$F_{\alpha\beta} \equiv -w \int^\Lambda \sum_{\rho \neq \alpha,\beta} \left(G(p) \Gamma^{(1)}(p) G(p) \right)_{\alpha\rho,\beta\rho}. \qquad (20)$$

In equations (19, 20) we use G instead of $\tilde{G}$, where G is defined like $\tilde{G}$ in equation (1), but with r and $Q_{\alpha\beta}^{(0)}$ replacing r_0 and $Q_{\alpha\beta}$, respectively. $Q_{\alpha\beta}^{(0)}$ is the leading term of the order parameter:

$$Q_{\alpha\beta} = Q_{\alpha\beta}^{(0)} + O(w^2).$$

The matrix elements of G^{-1} are now all zeroth order, nevertheless G is not a free propagator in the usual sense: since $x_1 \sim w^2$, all the sizes of the blocks in the hierarchical construction are also proportional to w^2 and thus G can be expanded in w^2:

$$G = G^{(0)} + G^{(1)} + \dots$$

As it turns out, $G^{(0)}$ is nothing but the near infrared propagator whose components were listed in reference [17] and further terms can be calculated by straightforward, although lengthy, perturbation methods (the component $G^{(1)}{}^{x_1,x}_{1,x_1}$ was needed for the present calculations). Two types of infrared divergences with different origin enter the equation of state:

– In equation (20), where $\Gamma^{(1)}$ is the first order contribution to the two-point vertex operator $\Gamma \equiv \hat{G}^{-1}$ whose zero momentum limit is the exact mass operator, the high infrared power of the two G's could lead to infinite terms. These are, however, compensated by the zero modes of the mass operator $\Gamma(0)$, as a direct consequence of exact Ward-identities (very much like in the case of the Heisenberg model). These Ward-identities follow from the permutation symmetry of the n replicas which becomes a continuous symmetry in the infinite step RSB limit [20]. Thus $F_{\alpha\beta}$ is innocent and well-behaved even below $d = 6$.

– Ward-identities cannot help us to get rid of infrared danger in $E_{\alpha\beta} = E(x)$ of equation (19). To illustrate the problem, we write down our result for $x = x_1$:

$$E(x_1) = wr^2 \int^{\Lambda} (g_1 g_2^2 - 2g_1^2 g_2) + w^3 r^3 \frac{4}{5} \left(\int^{\Lambda} g_1^2 g_2 \right)$$

$$\times \int^{\Lambda} (-4g_2^3 - 5g_1 g_2^2 - 28g_1^2 g_2 + 32g_1^3) + O(w^5), \tag{21}$$

where

$$g_1 \equiv \frac{1}{p^2 + r} \quad \text{and} \quad g_2 \equiv \frac{1}{p^2} .$$

All but one term in equation (21) give contributions for the computation of β (log's), checking of exponentiation (log^2's) and even to the amplitude C in equation (17) (simple powers) up to $O(\epsilon^2)$. The term $-4g_2^3$ is, however, infrared divergent, leading to an infinite contribution to C at order ϵ^2.

The origin of the infrared terms is identical to that in the mean field case: $\sum_{\rho \neq \alpha, \beta} G_{\alpha\rho, \beta\rho}(p)$ is p^{-3} in the zero momentum limit, but when expanding it in w^2, high infrared powers are generated order by order. These enter the amplitudes of scaling forms like equations (13, 16, 17), suggesting that amplitudes cannot be expanded without producing infinite terms.

The above scaling theory of the equation of state can be extended to the propagator $\hat{G}$ (a first attempt in this direction has been made in [22,23]). Indeed, it is plausible (and calculations on the Gaussian level suggest) that *any* component of $\hat{G}$ has the scaling form $p^{-(2-\eta)} g(p^2/r^{2\nu}, x, \dots)$ where the scaling function g is, of course, different for different components and ... stands for other possible overlaps of the given propagator component. We record here the explicit formula for the diagonal element $\hat{G}_{1,1}^{x,x}$ which is the Fourier transform of the overlap of the spin-spin correlation function (for the physical meaning of the propagators, see Ref. [17]):

$$G_{1,1}^{x,x}(p) = \frac{1}{p^{2-\eta}} g_{\text{diag}}(\frac{p^2}{r^{2\nu}}, x), \qquad d < 6. \tag{22}$$

For $x = 0$, it is also the most accessible to numerical simulations [8].

The theory presented above is based on the assumption of an ultrametrically organized Parisi RSB ansatz. Its analytical justification requires perturbation methods like the ϵ-expansion to compute critical exponents (η, ν and β) and check exponentiation. Amplitudes, however, are infected by infrared divergences, and the resummation of these infinities seems to be a hard task. The difficulties in using a perturbative approach for a system with a complex phase space structure has been discussed in the context of the random manifold problem in reference [21]. Numerical methods would be valuable to check formulae like

equations (14, 22), and compare the measured exponents with those obtained at or above T_c. For this purpose, we make further assumptions for $G_{1,1}^{x,x}$ in equation (22) when $x = 0$, namely

$$g_{\text{diag}}(v, 0) \rightarrow \text{constant}, \qquad v \rightarrow \infty,$$

$$g_{\text{diag}}(v, 0) \sim v^{-\frac{\lambda}{2}}, \qquad v \rightarrow 0. \tag{23}$$

The value of the exponent λ has an important role in the determination of the lower critical dimension d_l (see below).

Up to now, we confined our discussion to systems with $d < 8$. Above eight dimensions we must keep the bare quartic coupling u to avoid instability of the RS solution, and all the formulae containing $\tilde{u}$ are still valid with u replacing $\tilde{u}$. $Y(x)$ in equation (3) has the mean field form like in equation (11) with $\eta = 0$, $\nu = 1/2$ and $x_1 = ur/w^2 \sim r$. The scaling function f depends now on w, u, Λ and, in high enough dimension, possibly on other higher order bare couplings, and on the l.h.s of equation (3) terms like $\frac{2}{3} u Q_{\alpha\beta}^3$ appear. The theory is basically nonperturbative in the sense that a perturbation expansion in the bare couplings destroys the infrared cutoff at $p \sim r$.

The problem of the lower critical dimension d_l where the above scaling theory blows up, can be approached from two different directions:

– In the case of any critical transition, T_c can reach zero temperature in low enough dimension which, in a field theoretic representation, manifests itself in a divergent $r_0^{(c)}$. From the observation that the second term $Q_{\alpha\beta} r_0^{(c)}$ in $Y_{\alpha\beta}$ (see Eq. (3)) is to cancel the leading contribution of the first one when $p^2 \gg r$, we can put equation (5) into the simple form consistent with our scaling picture:

$$r_0^{(c)} \sim \lim_{r \to 0} r^{-\beta} \int^{\Lambda} \frac{1}{p^{2-\eta}} \tilde{f} \left(\frac{p^2}{r^{2\nu}} \right) \sim \int^{\Lambda} \frac{1}{p^{2-\eta+\beta/\nu}}, \tag{24}$$

where the asymptotic form $\tilde{f}(v) \sim v^{-\beta/2\nu}$, $v \rightarrow \infty$, has been used. This can now be compared with the subleading term of equation (16) remaining after the cancellation mentioned above. Using the scaling law of equation (15), we arrive at the conclusion that the integral of equation (24) is convergent provided $d > 2 - \eta$. The lower critical dimension following from this argument satisfies the equation

$$d_l^{(1)} = 2 - \eta(d_l^{(1)}). \tag{25}$$

– In systems with Goldstone modes, the high infrared powers of the propagators lead to divergences in local quantities at some $d_l^{(2)}$, thereby signalling the breakdown of the ordered phase. When $d_l^{(2)} > d_l^{(1)}$, it is the lower critical dimension (a situation known from the XY and Heisenberg models where $d_l = 2$). As it was suggested in reference [22], we must consider

the most dangerous component $G_{1,1}^{0,0}$ whose far infrared behaviour can be obtained from equations (22, 23):

$$G_{1,1}^{0,0}(p) \sim p^{-(2-\eta+\lambda)}, \qquad p \to 0. \qquad (26)$$

From this we have

$$d_1^{(2)} = 2 - \eta(d_1^{(2)}) + \lambda(d_1^{(2)}). \qquad (27)$$

In reference [22], after an incomplete perturbative analysis, it was argued that $\lambda = \beta/\nu$, giving rise to $d_1^{(2)} = d_1^{(1)}$, leading to a d_1 close to 2.5. Similar values were reached by Franz *et al.* [24] and Nifle and Hilhorst [25]. Given the degree of uncertainty in all these estimates, and the fact that corrections to the $1/p^3$ behaviour of other propagator components have not been investigated, the above results have to be taken with caution.

References

1. For a review see: M. Mézard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987); K. Binder, A.P. Young, Rev. Mod. Phys. **58**, 801 (1986).
2. S. Edwards, P.W. Anderson, J. Phys. F **5**, 965 (1975).
3. D. Sherrington, S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
4. G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979); J. Phys. A **13**, L115 (1980); **13**, 1101 (1980); **13**, 1887 (1980).
5. W.L. McMillan, J. Phys. C **17**, 3179 (1984).
6. D.S. Fisher, D. Huse, Phys. Rev. Lett. **56**, 1601 (1986); Phys. Rev. B **38**, 386 (1988).
7. A.J. Bray, M.A. Moore, in *Proceedings of the Heidelberg Colloquium on Glassy Dynamics*, Lecture Notes in Physics, 275, edited by J.L. van Hemmen, I. Morgenstern (Springer, Heidelberg, 1986); Phys. Rev. Lett. **58**, 57 (1987).
8. E. Marinari, G. Parisi, J.J. Ruiz-Lorenzo, in *Spin Glasses and Random Fields*, Series on Directions in Condensed Matter Physics, Vol. 12, edited by A.P. Young (World Scientific, 1998).
9. C.M. Newman, D.L. Stein, Phys. Rev. B **46**, 973 (1992); Phys. Rev. Lett. **76**, 515 (1996); Phys. Rev. Lett. **76**, 4821 (1996).
10. G. Parisi, cond-mat/9603101 (preprint); F. Guerra, Int. J. Mod. Phys. B **10**, 1675 (1996).
11. A. Georges, M. Mézard, J.S. Yedidia, Phys. Rev. Lett. **64**, 2937 (1990).
12. J.R.L. de Almeida, D.J. Thouless, J. Phys. A **11**, 983 (1978).
13. A.J. Bray, S.A. Roberts, J. Phys. C **13**, 5405 (1980).
14. O.F. de Alcantara Bonfim, J.E. Kirkham, A.J. McKane, J. Phys. A **14**, 2391 (1981); J.E. Green, J. Phys. A **17**, L43 (1985).
15. M.E. Ferrero, G. Parisi, J. Phys. A **29**, 3795 (1996).
16. M.E. Ferrero, G. Parisi, P. Ranieri, J. Phys. A **29**, L569 (1996).
17. C. De Dominicis, I. Kondor, T. Temesvári, in *Spin Glasses and Random Fields*, Series on Directions in Condensed Matter Physics, Vol. 12, edited by A.P. Young (World Scientific, 1998).
18. A.J. Bray, M.A. Moore, J. Phys. C **12**, 79 (1979).
19. J.E. Green, M.A. Moore, A.J. Bray, J. Phys. C **16**, L815 (1983).
20. C. De Dominicis, T. Temesvári, I. Kondor, J. Phys. IV France **8**, 13 (1998); T. Temesvári, I. Kondor, C. De Dominicis (to be published).
21. M. Mézard, G. Parisi, J. Phys. I France **2**, 2231 (1992).
22. C. De Dominicis, I. Kondor, T. Temesvári, Int. J. Mod. Phys. B **7**, 986 (1993).
23. D.M. Carlucci, Ph.D thesis, Pisa, March 1997.
24. S. Franz, G. Parisi, M.A. Virasoro, J. Phys. I France **4**, 1657 (1994).
25. M. Nifle, H.J. Hilhorst, Phys. Rev. Lett. **68**, 2992 (1992).